Anales del VII CONGRESO NACIONAL DE ESTUDIANTES DE POSTGRADO EN ECONOMÍA (CNEPE)

DEPARTAMENTO DE ECONOMÍA
UNIVERISDAD NACIONAL DEL SUR

INSTITUTO DE INVESTIGACIONES ECONÓMICAS Y SOCIALES DEL SUR (IIESS)
CONICET – UNIVERSIDAD NACIONAL DEL SUR

Bahía Blanca

Mayo de 2015

ISBN: 978-987-1648-39-9

Departamento de Economía

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Abstract. This paper extends the Ramsey-Cass-Koopmans growth model of optimal capital accumulation in discrete time by introducing a generic population growth law that satisfies the following properties: population is strictly increasing and bounded, and the population growth rate is decreasing to zero as time tends to infinity. We show that the optimization problem admits a unique solution that can be characterized by the Euler equation. A closed-form solution of the model is presented for the case of a Cobb-Douglas production function and a logarithmic utility function. In contrast to the original model, the solution is not always monotone.

Keywords. Economic growth; Ramsey model; discrete time; decreasing population growth rate; closed-form solution.

JEL classification: C62; O41

1 Introduction

In the original neoclassical model of economic growth due to Ramsey (1928) and extended by Cass (1965) and Koopmans (1963), it is assumed that the population grows at a constant rate $n > 0$. In discrete time it is natural to define this growth rate as:

$$n = \frac{L_{t+1} - L_t}{L_t}$$

where L_t is the population level at period t, which implies that

$$L_{t+1} = (1 + n)\, L_t.$$

Then, population grows exponentially, and for any initial population L_0, its level at time t is defined by:

$$L_t = L_0\, (1 + n)^t\,.$$

This assumption is plausible only for small values of t because growing exponentially, the population approaches infinity when t goes to infinity, which is clearly unrealistic. Verhulst (1838) considered that a stable population would have a characteristic saturation level, which is usually called the *carrying capacity* of the environment.[1] To incorporate this upper bound on the growth size, Verhulst introduced the logistic equation as an extension of the exponential model (see Geritz and Kisdi, 2004). Moreover, as described in Maynard (1974), a more realistic law of population growth should verify the following properties:

1. when population is small enough in proportion to environmental carrying capacity (denoted by L_∞), then it grows at a constant rate $n > 0$,
2. when population is large enough in proportion to L_∞, the economic resources become more scarce which affects negatively population growth,
3. population growth rate is decreasing to 0.

In discrete time, the logistic equation due to Pielou et al.(1969) and the Beverton-Holt equation (Beverton, 1957) are representative examples of population laws verifying these properties.

In this paper we extend the neoclassical economic growth model of Ramsey - Cass - Koopmans (hereafter the Ramsey model) in discrete time by assuming that population growth follows the properties defined above. The classic approach to study the Ramsey model is through the value function and the Bellman equation (Lucas, Stokey, and Prescott, 1989). Since the

[1] In Maler, Perrings, and Pimentel (1995) the reader can find detailed information about the concept of carrying capacity of human population.

population growth rate in the extended version of the Ramsey model is exogenous and non-constant, it cannot be used a recursive method. Indeed, the optimization problem is not equivalent to a single functional equation (the Bellman equation). Thus, in this paper we explore a different approach to solve the optimization problem that emerges in the Ramsey model. To the best of our knowledge, this is the first paper that studies the Ramsey model in discrete time and where population size tends to a finite saturation level in the long-run. As we will note, all previous articles that analyze similar problems, focus in the Ramsey model but in continuous time.

A handful other papers have extended some classic economic growth models by considering a population growth hypothesis different from the exponential law. In the framework of the economic growth model due to Solow, Accinelli and Brida (2007a), Brida (2008), Donghan (1998) study a general model with a population growth rate that satisfies the previous properties. Brida and Pereyra (2008) solves the same problem but in discrete time. Moreover, Brida and Maldonado (2010), Guerrini (2006), Guerrini (2011), and Ferrara (2011b) find a closed-form solution for an economy with a Cobb-Douglas production function, and study its properties.[2]

Regarding to the Ramsey model, Guerrini (2010c) analyzes a general model with a bounded population growth rate in continuous time. Accinelli and Brida (2007b) suppose a logistic population growth, and solve the model with a general production function. Assuming a Cobb-Douglas production function, Guerrini (2009) studies the case where the population follows a logistic law, while Guerrini (2010a) and Guerrini (2010b) use the Von Bertalanffy population law.[3]

Finally, since the main question of the paper has been studied in continuous time, it is worth noting that its extension to discrete time is innovative and presents new difficulties. Indeed, the choice between discrete versus continuous time can affect dramatically the outcomes of a model and conclusions that might draw from it, because dynamics of the two types of models can be completely different and lead to different predictions.[4] In economic modeling both types of timing are present and there does not exist a common view between economists on which representation of time is better to model in economics.[5] This gives as a first challenge the study of a discrete version of the Ramsey model with decreasing population growth rate, since previous articles that study the same problem focus on the continuous version. Moreover, when one changes a model from continuous to discrete time the complexity of dynamics increases. Then, another challenge to study the discrete version of the Ramsey model is technical: we have a different mathematical object

[2]See also Ferrara (2011c), Ferrara (2011a), Bay (2013), and Michetti (2013).

[3]See Guerrini (2010d), Guerrini (2010e), Ferrara and Guerrini (2009), and Ferrara and Guerrini (2011) for other interesting cases.

[4]One of the main examples of a model that is rather different when we change timing is the logistic population law.

[5]See Medio (1991) for a discussion of time dimension in economic models and references therein.

to study and then innovative techniques must be introduced.

2 The Ramsey model in discrete time

The standard version of the model considers an economy that produces a unique good that can be consumed or used (along with homogeneous labour) as capital in the production. The economy is endowed with a technology defined by a production function $F(K, L)$, where K is the capital stock and L the level of population, with the following properties:

- $\frac{\partial F(K,L)}{\partial K} > 0$, $\frac{\partial F(K,L)}{\partial L} > 0$, $\frac{\partial^2 F(K,L)}{\partial K^2} < 0$, $\frac{\partial^2 F(K,L)}{\partial L^2} < 0$.
- $F(K, 0) = F(0, L) = 0; \forall K, L \in R^+$.
- $F(\lambda K, \lambda L) = \lambda F(K, L); \forall \lambda, K, L \in R^+$(constant return to scale).
- $\lim\limits_{K \longrightarrow 0} \frac{\partial F(K,L)}{\partial K} = \lim\limits_{L \longrightarrow 0} \frac{\partial F(K,L)}{\partial L} = +\infty$.
- $\lim\limits_{K \longrightarrow +\infty} \frac{\partial F(K,L)}{\partial K} = \lim\limits_{L \longrightarrow +\infty} \frac{\partial F(K,L)}{\partial L} = 0$ (INADA conditions).

Production may be consumed or accumulated as capital for future production:

$$F(K_t, L_t) = C_t + K_{t+1} - (1 - \delta)K_t \tag{1}$$

where $\delta \in (0, 1)$ denotes the capital depreciation rate. F can be expressed in per capita terms as:

$$\frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = g(k_t)$$

where g is the production function expressed in its intensive form, and $k_t = \frac{K_t}{L_t}$. Since the number of workers equals the number of consumers and has a growth rate of n, equation (1) can be rewritten as:

$$g(k_t) = c_t + k_{t+1}(1 + n) - (1 - \delta)k_t$$

Now, define $f(k_t) = g(k_t) + (1 - \delta)k_t$. It is straightforward to prove that f has the following properties:

- $f(0) = 0$
- $f'(k) > 0 \quad \forall k$
- $f''(k) < 0 \quad \forall k$
- $\lim\limits_{k \to 0} f'(k) = \infty$

The economy has a representative consumer with an utility function $u : R_+ \to R$ such that: $u' > 0$ and $u'' < 0$ (strictly increasing and strictly concave). There is a social planner who decides, in each period, how to divide the production between consumption and investment, such that the population achieves the maximum utility level. Denote by β the representative consumer's discount factor. Thus, the social planner faces the following problem:

$$\begin{cases} \text{Max} \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - (1+n)k_{t+1}) \\ s.t: \\ 0 \leq k_{t+1} \leq \dfrac{f(k_t)}{1+n} \\ \text{With } k_0 > 0 \text{ given.} \end{cases}$$

The problem is usually solved by standard tools of dynamic programming. In particular, the value function and the Bellman equation are employed to show that there exists a unique solution to the problem that converges monotonically to a value k_∞ defined by the equation $f'(k_\infty) = \frac{1+n}{\beta}$.[6] We finish this section with the following remark that compares the behavior of the solutions of two economies that differ only in the rate of population growth (we omit the proof since it is straightforward).

Remark 1. Consider an economy with a rate of population growth n^0, and another economy with the same fundamentals but with a rate of population growth n^1, such that $n^1 > n^0$. Denote by $\{k_t^0\}_t$ and $\{k_t^1\}_t$ ($\{c_t^0\}_t$ and $\{c_t^1\}_t$), the optimal capital (consumption) sequences of each economy, and k_∞^0, k_∞^1 (c_∞^0, c_∞^1), the optimal capital (consumption) value in the steady state of each economy.

Then, there exists T such that $k_t^0 \geq k_t^1$ for all $t \geq T$, and $k_\infty^0 \geq k_\infty^1$. Moreover, $c_t^0 \geq c_t^1$ for all $t \geq T$, and $c_\infty^0 \geq c_\infty^1$.

The remark implies that an economy with a population growth rate lower than other economy with the same fundamentals, has a higher capital level in its steady state, and after some initial periods, the capital of the first economy is higher than the capital of the second economy. Also, the long term per capita consumption is higher in the economy with a rate of population growth of n_0 than in the economy with n_1.

3 The Ramsey model with decreasing population growth rate

When the rate of population growth is not constant over time, the approach of the Bellman equation cannot be applied. Also, the difference equation

[6] A detailed presentation of these results can be found in Lucas, Stokey, and Prescott, (1989).

that arises in the optimization problem is non autonomous, which prevents the use of standard techniques. In this section, we follow the formulation of Le Van and Dana (2003), and we extend their analysis to the case where the population growth rate is not-constant and decreasing to zero. The last part of this section, presents a closed-form solution of the model for the case of a Cobb-Douglas production function and a logarithmic utility function.

Assume that the population evolves according to the following equation:

$$L_{t+1} = (1+n_t)L_t$$

where n_t is the population growth rate, and satisfies:

$$0 \leq n_{t+1} \leq n_{t,} \forall t \in N, \ \lim_{t \longrightarrow +\infty} n_t = 0$$

with L_0 (given) the initial population level.

In this framework, the problem faced by the social planner is:

$$(P) \left\{ \begin{array}{l} \text{Max} \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - (1+n_t)k_{t+1}) \\ s.t: \\ 0 \leq k_{t+1} \leq \dfrac{f(k_t)}{1+n_t} \\ 0 \leq n_{t+1} \leq n_t \\ \lim\limits_{t \longrightarrow +\infty} n_t = 0 \end{array} \right.$$

with $n_0 > 0$ and $k_0 > 0$ given.

3.1 Existence of optimal paths and properties

First, we will show that problem (P) is equivalent to the maximization of a continuous function defined on a compact set, and thus, the problem has a solution.

Since:

$$\lim_{k \to +\infty} f'(k) = 1 - \delta < 1, \text{ and } \lim_{k \to 0} f'(k) = +\infty,$$

we know that there exists $\overline{k} > 0$ such that $f(\overline{k}) = \overline{k}$, and $f(k) > k \ \ \forall k < \overline{k}$ and $f(k) < k \ \ \forall k > \overline{k}$. Moreover, it is easy to show that given a feasible sequence $\{k_t\}_t$, that is, a sequence such that $0 \leq k_{t+1} \leq \dfrac{f(k_t)}{1+n_t}$ $\forall t \geq 0$ with $k_0 > 0$, it holds that:

$$0 \leq k_t \leq \max \left\{ k_0, \overline{k} \right\} = \lambda, \ \forall t \geq 0$$

Also:

$$0 \leq c_t \leq f(k_t) \leq f(\lambda) \ \forall t \geq 0$$

and since u is a continuous function, u is a bounded function in $[0, f(\lambda)]$.

Denote as $\pi(k_0)$ the set of feasible sequences from k_0.[7]

We know that if $\{k_t\}_t \in \pi(k_0)$, then $\{k_t\}_t \in [0,\lambda]^{+\infty}$. Since $[0,\lambda]^{+\infty}$ is a compact set for the product topology defined in the space of sequences, $\{k_t\}_t, \pi(k_0)$ is included in a compact set. Moreover, as $\pi(k_0)$ is closed, it is compact.

Lemma 1. *$\pi(k_0)$ is a compact set for the product topology.*

Proof. Let $(k_0, k_1^m, k_2^m, ..., k_t^m, ...)$ be a sequence of feasible sequences that converges to $(k_0, k_1, k_2, \dots, \dots, k_t, ...)$. By the definition of the product topology we have: $k_t^m \underset{m\to+\infty}{\longrightarrow} k_t \ \forall t$. Since each sequence is feasible $0 \leq k_{t+1}^m \leq \frac{f(k_t^m)}{1+n_t} \forall t$, and computing the limit when $m \to +\infty$: $0 \leq k_{t+1} \leq \frac{f(k_t)}{1+n_t}$. Thus, $\pi(k_0)$ is closed. □

Define the function $U : \pi(k_0) \to R$ such that: $U(k) \equiv U(\{k_t\}_t) = \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - (1+n_t)k_{t+1})$.

Since u is bounded, U is well-defined. In particular, note that:

$$\left|\sum_{t=0}^{+\infty} \beta^t u(f(k_t) - (1+n_t)k_{t+1})\right| \leq \sum_{t=0}^{+\infty} \beta^t \left|u(f(\lambda))\right| = \left|\frac{u(f(\lambda))}{1-\beta}\right|$$

Lemma 2. *U is a continuous function.*

Proof. See Appendix for a proof. □

Thus, by the previous lemmas, (P) can be written as the problem of finding the maximum of a continuous function defined in a compact set.

Proposition 1. *(P) is equivalent to the following problem:*

$$(P') \left\{ \begin{array}{l} Max\ U(k) \\ s.t: \\ k \in \pi(k_0) \end{array} \right.$$

Moreover, (P) has a solution, that is, there exists $\widehat{k} \in \pi(k_0)$ such that $U(\widehat{k}) \geq U(k)$ for all $k \in \pi(k_0)$.

The next proposition shows that (P) has a unique solution. Next, we prove that all the elements of the optimal sequence are positive, and also that the solution satisfies a version of the Euler equation adapted to our framework.

Proposition 2. *Given $k_0 > 0$, there exists a unique solution to (P) and a unique optimal sequence of consumption.*

[7] We will also called a sequence $\{c_t\}_t$ feasible if $c_t = f(k_t) - (1+n_t)k_{t+1}$ for some feasible sequence $\{k_t\}_t$.

Proof. See Appendix for a proof. □

Proposition 3. *The optimal sequence of capital and per capita consumption verify that $k_t > 0$ and $c_t > 0$ for all t.*

Proof. See Appendix for a proof. □

Proposition 4. *If $k_0 > 0$ the optimal capital sequence satisfies the Euler equation:*

$$\frac{u'(f(k_t) - (1+n_t)k_{t+1})}{\beta u'(f(k_{t+1}) - (1+n_{t+1})k_{t+2})} = \frac{f'(k_{t+1})}{1+n_t}$$

Proof. See Appendix for a proof. □

The Euler equation implies that, in the optimal path, the relation between the marginal utility of consumption in one period and the marginal utility of consumption in the following period, should be equal to the rate under which the economy transforms goods from one period to the following. Although it is necessary, the Euler equation is not sufficient; in order a sequence to be optimal it should verify an additional condition: the tranversality condition. This result is known as the Mangasarian lemma.

Proposition 5. *Let $\{k_t\}_{t=0}$ be a feasible sequence that satisfies the Euler equation and the following tranversality condition:*

$$\lim_{T\to+\infty} \beta^T u'(f(k_T) - (1+n_T)k_{T+1})k_{T+1}(1+n_T) = 0$$

Then, the sequence is optimal for the problem (P).

Proof. See Appendix for a proof. □

3.2 A closed-form solution

To study the convergence of the optimal sequence we consider an economy with a logarithmic utility function, and a Cobb-Douglas production function. Thus, assume that:

$$u(c_t) = \log(c_t)$$
$$f(k_t) = k_t^\alpha \text{ with } \alpha \in (0,1)$$

The next theorem presents a closed-form solution of the problem (P). In particular we will prove that the sequence $\{k_t\}_{t=0}$ such that $k_{t+1} = \frac{\alpha\beta}{1+n_t}k_t^\alpha$ is the solution of problem (P) and converges.

Theorem 1. *Consider the Ramsey model with a decreasing population growth rate, a logarithmic utility function, and a Cobb-Douglas production function. Then, the optimal capital sequence $\{k_t\}_{t=0}$ is such that,*

$$k_{t+1} = \frac{\alpha\beta}{1+n_t}k_t^\alpha, \forall t$$

and converges to $k_\infty = (\alpha\beta)^{\frac{1}{1-\alpha}}$.

Proof. The proof consists in showing that $\{k_t\}_{t=0}$ satisfies the Euler equation and the Mangasarian lemma.

1. We will prove that the sequence $\{k_t\}_{t=0}$ satisfies the Euler equation. Note that:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{f(k_{t+1}) - (1+n_{t+1})k_{t+2}}{f(k_t) - (1+n_t)k_{t+1}} = \frac{f(k_{t+1}) - \alpha\beta k_{t+1}^{\alpha}}{f(k_t) - \alpha\beta k_t^{\alpha}}$$

Considering that $f(k_t) = k_t^{\alpha}$ we have,

$$\frac{f(k_{t+1}) - \alpha\beta k_{t+1}^{\alpha}}{f(k_t) - \alpha\beta k_t^{\alpha}} = \frac{k_{t+1}^{\alpha} - \alpha\beta k_{t+1}^{\alpha}}{k_t^{\alpha} - \alpha\beta k_t^{\alpha}} = \frac{k_{t+1}^{\alpha}}{k_t^{\alpha}} = \left(\frac{\alpha\beta}{1+n_t}k_t^{\alpha}\right)^{\alpha}\frac{1}{k_t^{\alpha}}$$

which implies that:

$$\left(\frac{\alpha\beta}{1+n_t}\right)^{\alpha} k_t^{\alpha(\alpha-1)} = \frac{\beta}{1+n_t}\alpha\left(\frac{\alpha\beta}{1+n_t}k_t^{\alpha}\right)^{\alpha-1} =$$

$$\frac{\beta}{1+n_t}\alpha k_{t+1}^{\alpha-1} = \frac{\beta}{1+n_t}f'(k_{t+1}).$$

2. Compute:

$$\lim_{T\to+\infty}\beta^T u'(f(k_t) - (1+n_T)k_{T+1})(1+n_T)k_{T+1} =$$

$$\lim_{T\to+\infty}\beta^T\frac{1}{k_T^{\alpha}(1-\alpha\beta)}(1+n_T)\frac{\alpha\beta}{1+n_T}k_T^{\alpha} =$$

$$\lim_{T\to+\infty}\beta^T\frac{\alpha\beta}{1-\alpha\beta} = 0$$

since $\beta \in (0,1)$.

3. Finally, we will prove that the optimal sequence converges. If $k_{t+1} = \frac{\alpha\beta}{1+n_t}k_t^{\alpha}$ y $k_0>0$ then:

$$k_t = \left(\frac{\alpha\beta}{1+n_{t-1}}\right)\left(\frac{\alpha\beta}{1+n_{t-2}}\right)^{\alpha}\left(\frac{\alpha\beta}{1+n_{t-3}}\right)^{\alpha^2}\cdots\cdots\left(\frac{\alpha\beta}{1+n_0}\right)^{\alpha^{t-1}} k_0^{\alpha^t}$$

$$k_t = \frac{(\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}}}{(1+n_{t-1})(1+n_{t-2})^{\alpha}(1+n_{t-2})^{\alpha^2}....(1+n_0)^{\alpha^{t-1}}}k_0^{\alpha^t}$$

Taking logs:

$$\log(k_t) = \left(\frac{1-\alpha^t}{1-\alpha}\right)\log(\alpha\beta) + \alpha^t\log(k_0) -$$

$$[\log(1+n_{t-1}) + \alpha\log(1+n_{t-2}) + ... + \alpha^{t-1}\log(1+n_0)]$$

Define $a_t = \log(1+n_{t-1}) + \alpha\log(1+n_{t-2}) + ... + \alpha^{t-1}\log(1+n_0)$.

In the Appendix it is shown that a_t converges to zero.

Given that $\log(k_t) = (\frac{1-\alpha^t}{1-\alpha})\log(\alpha\beta) + \alpha^t\log(k_0) - a_t$, and $\alpha \in (0,1)$, we have:

$$\lim_{t\to+\infty}\log(k_t) = \frac{1}{1-\alpha}\log(\alpha\beta)$$

Thus, $\{k_t\}_t$ converges to $k_\infty = (\alpha\beta)^{\frac{1}{1-\alpha}}$, which is the same value that it is found in the original Ramsey model with $n = 0$. □

Remark 2. The closed-form solution that we found implies directly the following results.

1. If $k_0 < k_\infty$, the optimal capital sequence is increasing.
2. If $k_0 \geq k_\infty$, there are some values of the parameters of the model such that the optimal capital sequence is not monotone. In particular, if $k_0 = k_\infty$, the optimal sequence decreases during the initial periods, and then increases. This result contrasts with the conclusion in the classic model where the optimal sequence is monotone for all possible values of the parameters.
3. Let $\{k_t^1\}_t$ be the optimal capital sequence of an economy with a rate of population growth defined by $\{n_t^1\}_t$, and $\{k_t^2\}_t$ the optimal capital sequence in other economy with a rate of population growth defined by $\{n_t^2\}_t$. If $n_t^1 < n_t^2$ for each t, then $k_t^1 > k_t^2$ for each t. Thus, an economy with a rate of population growth smaller in each period than other, has in each period a higher capital level (in per capita terms).
4. The solution of the model with initial condition k_0 is asymptotically stable, which implies that small variations of the initial capital do not have large impacts on the economic growth process.
5. Finally, note that the general solution does not depend on the particular form of $\{n_t\}_t$.

4 Conclusions

In growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because, in particular, it implies that population goes to infinity when time goes to infinity. In this paper we suggest a more realistic approach by considering in the framework of the Ramsey model that population is strictly increasing and bounded, and that its rate of growth is strictly decreasing to zero. In the extended model, the classic approach that uses the Bellman equation does not apply, and we introduce other tools to study the model.

The paper shows that there exists a unique solution of the model that satisfies the Euler equation. Moreover, if the utility is logarithmic and the production function is Cobb-Douglas, the optimal sequence converges to $k_\infty = (\alpha\beta)^{\frac{1}{1-\alpha}}$, and then it depends only on the technology f, the rate of capital depreciation δ, the discount factor β and thus, the intrinsic rate of population growth n_t plays no role in determining the long run level of per capita output.

When the initial capital level is lower than its level in the steady state, the optimal capital sequence is increasing, as in the original model. However, there are some values of the initial capital level such that the sequence is not monotone, and in particular, there is no constant solution to the maximization problem.

Many open questions remain after our analysis. First, the same analysis conducted to obtain a closed-form solution may be redone but considering other utility and production functions, and then, the new results may be compared with those obtained in this paper. Second, it would be interesting to consider a particular law for the population growth, for instance, the Beverton-Holt equation. In this case, the problem will involve an autonomous difference equation, and then classic approaches can be applied. Third, as a future research, the behavior of optimal sequences for different values of the parameters of the model may be studied, this analysis will complement the main findings of this paper. Finally, if it is assumed that $n_{t+1} < n_t, \forall t \geq 0$ and $\lim\limits_{t\to+\infty} n(t) = \widetilde{n} > 0$, the same results with suitable modifications may be obtained. But this is material of future research.

5 Appendix

5.1 Proof of Lemma 2

Let $(k_0, k_1^m, k_2^m, \ldots, k_t^m, \ldots)$ be a sequence of feasible sequences that converges to $(k_0, k_1, k_2, \ldots, k_t, \ldots)$. Then we have:

$$|U(k^m) - U(k)| \leq$$

$$\sum_{t=0}^{+\infty} \beta^t \left| u(f(k_t) - (1+n_t)k_{t+1}) - (u(f(k_t^m) - (1+n_t)k_{t+1}^m)) \right|$$

Let $\epsilon > 0$. We know that $U(k^m)$ and $U(k)$ converge, then there exists T such that for all m:

$0 \leq \sum\limits_{t=T}^{+\infty} \beta^t u(f(k_t) - (1+n_t)k_{t+1}) \leq \frac{\epsilon}{3}$

$0 \leq \sum\limits_{t=T}^{+\infty} \beta^t u(f(k_t^m) - (1+n_t)k_{t+1}^m) \leq \frac{\epsilon}{3}$.

Thus,

$$|U(k^m) - U(k)| \leq$$

$$\sum_{t=0}^{T}\beta^t \left|u(f(k_t)-(1+n_t)k_{t+1}-(u(f(k_t^m)-(1+n_t)k_{t+1}^m)\right|+\frac{2}{3}\epsilon$$

Since u and f are continuous functions, and $k_t^m \underset{m\to+\infty}{\to} k_t$, there exists N such that for all $m \geq N$:

$\sum\limits_{t=0}^{T}\beta^t \left|u(f(k_t)-(1+n_t)k_{t+1}-u(f(k_t^m)-(1+n_t)k_{t+1}^m)\right| < \frac{\epsilon}{3}$.

Thus, for all $m \geq N$, $|U(k^m)-U(k)| < \epsilon$, which implies that U is continuous.

5.2 Proof of Proposition 2

Proof. Suppose there exist two different optimal solutions k^1 and k^2, and let c^1, c^2 be the associated consumption sequences. Thus for all t we have:

$$c_t^1+(1+n_t)k_{t+1}^1 = f(k_t^1)$$
$$c_t^2+(1+n_t)k_{t+1}^2 = f(k_t^2)$$

It must exist t such that $c_t^1 \neq c_t^2$ (otherwise, $k^1 = k^2$).

Let $\lambda \in (0,1)$, for all t we have that:

$$\lambda c_t^1+(1-\lambda)c_t^2+\lambda(1+n_t)k_{t+1}^1+(1-\lambda)(1+n_t)k_{t+1}^2 = \lambda f(k_t^1)+(1-\lambda)f(k_t^2)$$

and since f is concave $\lambda f(k_t^1)+(1-\lambda)f(k_t^2) \leq f(\lambda k_t^1+(1-\lambda)k_t^2)$

Note that:

$$0 \leq \lambda k_t^1+(1-\lambda)k_t^2 \leq \lambda\frac{f(k_t^1)}{1+n_t}+(1-\lambda)\frac{f(k_t^2)}{1+n_t} \leq \frac{f(\lambda k_t^1+(1-\lambda)k_t^2)}{1+n_t}.$$

Then, $\left\{\lambda k_t^1+(1-\lambda)k_t^2\right\}_{t=0}$ is a feasible sequence from k_0.

Consider $\sum\limits_{t=0}^{+\infty}\beta^t u(f(\lambda k_t^1+(1-\lambda)k_t^2)-(1+n_t)(\lambda k_{t+1}^1+(1-\lambda)k_{t+1}^2)) \geq \sum\limits_{t=0}^{+\infty}\beta^t u(\lambda c_t^1+(1-\lambda)c_t^2) \geq \sum\limits_{t=0}^{+\infty}\beta^t(\lambda u(c_t^1)+(1-\lambda)u(c_t^2))$, since u is concave and increasing.

$\sum\limits_{t=0}^{+\infty}\beta^t(\lambda u(c_t^1)+(1-\lambda)u(c_t^2)) = \sum\limits_{t=0}^{+\infty}\beta^t u(c_t^1)$, since $\{c_t^1\}_t$ is an optimal sequence.

Since there exists t such that $c_t^1 \neq c_t^2$, $u(\lambda c_t^1+(1-\lambda)c_t^2) > \lambda u(c_t^1)+(1-\lambda)u(c_t^2)$.

Then, $\sum\limits_{t=0}^{+\infty}\beta^t(\lambda u(c_t^1)+(1-\lambda)u(c_t^2)) > \lambda\sum\limits_{t=0}^{+\infty}\beta^t\mu(c_t^1)+(1-\lambda)\sum\limits_{t=0}^{+\infty}\beta^t u(c_t^2) = \sum\limits_{t=0}^{+\infty}\beta^t u(c_t^1)$.

Finally, we have that $\{c_t^1\}_t$ is not an optimal sequence, which is a contradiction. □

5.3 Proof of Proposition 3

Proof. Since $k_0 > 0$, the sequence with all its elements equal zero is not optimal, then there exists t, such that $c_t > 0$. For the sake of simplicity, assume that $c_0 = 0$ and $c_1 > 0$. Then, $(1+n_0)k_1 = f(k_0)$, $c_1 + (1+n_1)k_2 = f(k_1)$. Since $k_0 > 0$, $k_1 = \frac{f(k_0)}{1+n_0} > 0$.

Let $\epsilon > 0$ such that $k_1 - \epsilon > 0$ and $f(k_1 - \epsilon) - (1+n_0)k_2 > 0$.[8]

Define the following sequence:

$c_0^1 = \epsilon$,

$c_1^1 = f(k_1 - \epsilon) - (1+n_1)k_2$,

and $\forall t \geq 2 \quad c_t^1 = c_t \quad$ and $k_t^1 = k_t$. The sequences c_t^1 and k_t^1 are feasible from k_0.

Let compute $\Delta\epsilon = \sum_{t=0}^{+\infty} \beta^t u(c_t^1) - \sum_{t=0}^{+\infty} \beta^t u(c_t) = u(c_0^1) + \beta u(c_1^1) - (u(c_0) + \beta u(c_1)) = u(c_0^1) - u(c_0) + \beta u(c_1^1) - \beta u(c_1) = u(c_0^1) - u(c_0) + \beta(u(c_1^1) - \beta u(c_1))$.

Since u is a concave function, we have:

$u'(\epsilon)(x-\epsilon) + u(\epsilon) \geq u(x) \quad \forall x \geq 0, \forall \epsilon$, then $u'(\epsilon)(x-\epsilon) \geq u(x) - u(\epsilon)$ and taking $x = 0$:

$$u'(\epsilon)\epsilon \leq u(\epsilon)$$

which implies:

$$\begin{cases} u(c_0^1) - u(c_0) \geq u'(c_0^1)(c_0^1 - c_0) = u'(c_0^1)\epsilon \\ u(c_1^1) - u(c_1) \geq u'(c_1^1)(c_1^1 - c_1) = u'(c_1^1)(f(k_1 - \epsilon) - f(k_1)) \end{cases}$$

Then,

$$\begin{aligned} \Delta\epsilon &\geq u'(c_0^1)\epsilon + \beta\left[u'(c_1^1)(f(k_1-\epsilon) - f(k_1))\right] \\ &= \epsilon\left[u'(c_0^1) + \beta u'(c_1^1)\left(\frac{f(k_1-\epsilon) - f(k_1)}{\epsilon}\right)\right] \end{aligned}$$

Given that:

$$u'(c_0^1) \underset{\epsilon \longrightarrow 0}{\longrightarrow} \infty \quad (\ u'(0) = +\infty)$$

and

$$u'(c_1^1)\left(\frac{f(k_1-\epsilon) - f(k_1)}{\epsilon}\right) \underset{\epsilon \longrightarrow 0}{\to} -\ u'(c_1^1) f'(k_1) > -\infty$$

For ϵ small enough we have: $\Delta\epsilon > 0$, which means that the total utility (in present value) is higher when $c_0^1 = \epsilon$ than when $c_0 = 0$. Then, $c_0 > 0$.

Now, we will prove that $k_1 > 0$. By way of contradiction, suppose $k_1 = 0$. Since it must hold that $0 \leq k_2 \leq \frac{f(k_1)}{1+n_1} = 0$, we have $k_2 = 0$, and by induction: $k_t = 0$ and $c_t = 0 \ \forall t \geq 1$.

[8] Such ϵ exists since $k_1 > 0$ and $c_1 = f(k_1 - \epsilon) - (1+n_0)k_2 > 0$

As before, define the sequences c^1 and k^1 by:

$$c_0^1 = c_0 - \epsilon k_1^1 = \frac{\epsilon}{1+n_0}$$

Since $c_0 > 0$, there exists ϵ such that $c_0 - \epsilon > 0$.
Then, $c_1^1 = f(\frac{\epsilon}{1+n_0})$ and $\forall t \geq 2$ $c_t^1 = c_t$ and $k_t^1 = k_t$.
Thus,

$$\Delta\epsilon = \sum_{t=0}^{+\infty} \beta^t u(c_t) - \sum_{t=0}^{+\infty} \beta^t u(c_t^1) = u(c_0) - u(c_0^1) + \beta \left[u(c_1) - u(c_1^1)\right] \geq$$

$$u'(c_0 - \epsilon)(-\epsilon) + \beta u'(f(\frac{\epsilon}{1+n_0}))f(\frac{\epsilon}{1+n_0}) =$$

$$\epsilon \left[\frac{\beta u'(f(\frac{\epsilon}{1+n_0}))f(\frac{\epsilon}{1+n_0})}{\epsilon} - u'(c_0 - \epsilon) \right]$$

Given that $\lim\limits_{\epsilon \longrightarrow 0} \frac{u'(f(\frac{\epsilon}{1+n_0}))f(\frac{\epsilon}{1+n_0})}{\epsilon} = +\infty$ and $\lim\limits_{\epsilon \longrightarrow 0} u'(c_0 - \epsilon) = u'(c_0) < \infty$. Then $\Delta\epsilon > 0$ for $\epsilon > 0$ small enough, which implies $k_1 > 0$.

Then, if $c_0 > 0$ it holds that $k_1 > 0$. Since $k_1 > 0$, by the same reasoning it can be shown that $c_1 > 0$, and if $c_1 > 0$, then $k_2 > 0$. Finally, by induction $c_t > 0$ and $k_t > 0$ $\forall t$. □

5.4 Proof of Proposition 4

Proof. From previous results we know that an optimal sequence satisfies:

$$0 < k_{t+1} < \frac{f(k_t)}{1+n_t} \quad \forall t \geq 0$$

and

$$c_t > 0, k_t > 0 \forall t$$

Let consider the sequence k^1 defined by:

$$k_T^1 = k_t,\ k_{T+1}^1 = y \ \forall T = t+1$$

with y defined in an open neighborhood of k_{t+1} such that:

$$\begin{cases} 0 < y(1+n_t) < f(k_t) \\ 0 < k_{t+2}(1+n_{t+1}) < f(y) \end{cases}$$

this means that k^1 is feasible from k_0.
Given that k is optimal, it holds that

$$U(k) \geq U(k^1),$$

which implies that

$$u(f(k_t) - (1+n_t)k_{t+1}) + \beta u(f(k_{t+1}) - (1+n_{t+1})k_{t+2}) \geq$$

$$\geq u(f(k_t) - y(1+n_t)) + \beta u(f(y) - (1+n_{t+1})k_{t+2})$$

This inequality holds for all y in an open neighborhood of k_{t+1}, and thus the function

$$\phi(y) = u(f(k_t) - y(1+n_t)) + \beta u(f(y) - (1+n_{t+1})k_{t+2})$$

has a local minimum at k_{t+1}. Thus, $\phi'(k_{t+1}) = 0$.

But,

$$\phi'(k_{t+1}) = u'(f(k_t) - (1+n_t)k_{t+1})(-(1+n_t)) + \beta u'(f(k_{t+1}) -$$

$$(1+n_{t+1})k_{t+2})f'(k_{t+1}) = 0$$

Which implies that:

$$\frac{u'(f(k_t) - (1+n_t)k_{t+1})}{\beta u'(f(k_{t+1}) - (1+n_{t+1})k_{t+2})} = \frac{f'(k_{t+1})}{1+n_t}$$

□

5.5 Proof of Proposition 5

Let $\{k_t\}_{t=0}$ be a feasible sequence that satisfies the Euler equation and $\{k_t^1\}_{t=0}$ other feasible sequence.

Define:

$$\Delta T = \sum_{t=0}^{T} \beta^t u(f(k_t) - (1+n_t)k_{t+1}) - \sum_{t=0}^{T} \beta^t u(f(k_t^1) - (1+n_t)k_{t+1}^1)$$

Compute:

$$\Delta T = u(f(k_0) - (1+n_0)k_1) - u(f(k_0) - (1+n_0)k_1^1) + \beta[u(f(k_1) - (1+n_1)k_2) -$$

$$u(f(k_1^1) - (1+n_1)k_2^1) + \ldots$$

$$+\beta^T[u(f(k_T) - (1+n_T)k_{T+1}) - u(f(k_T^1) - (1+n_T)k_{T+1}^1)]$$

u is concave, so it holds that:[9]

$$u(f(k_0) - (1+n_0)k_1) - u(f(k_0) - (1+n_1)k_1^1) \geq$$

$$u'(f(k_0) - (1+n_0)k_1)(1+n_0)(k_1^1 - k_1)$$

[9]Note that if u is concave and differentiable, then: $u(x) \leq u(a) + u'(a)(x-a)$ and $u(a) - u(x) \geq u'(a)(a-x)$ for all a and x in the domain of u.

and by the Euler equation:

$$u(f(k_0) - (1+n_0)k_1) - u(f(k_0) - (1+n_t)k_1^1) \geq$$
$$\beta f'(k_1)u'(f(k_1 - (1+n_1)k_2)(k_1^1 - k_1)$$

Thus,

$$\Delta T \geq \beta f'(k_1)u'(f(k_1) - (1+n_1)k_2)(k_1^1 - k_1)+$$
$$\beta[u(f(k_1) - (1+n_1)k_2) - u(f(k_1^1) - (1+n_1)k_2^1) + \ldots +$$
$$\beta^T[u(f(k_T) - (1+n_T)k_{T+1}) - u(f(k_T^1) - (1+n_T)k_{T+1}^1)]$$

Once again, given that u is concave:

$$u(f(k_1) - (1+n_1)k_2) - u(f(k_1^1) - (1+n_1)k_2^1) \geq$$

u'(f(k$_1$) $- (1+n_1)k_2)[f(k_1) - f(k_1^1) + (1+n_1)(k_2^1 - k_2)]$

Now, by the concavity of f, we know:

$$f(k_1) - f(k_1^1) \geq f'(k_1)(k_1 - k_1^1),$$

and then:

$$u(f(k_1) - (1+n_1)k_2) - u(f(k_1^1) - (1+n_1)k_2^1) \geq$$
$$u'(f(k_1) - (1+n_1)k_2)[f'(k_1)(k_1 - k_1^1) + (1+n_1)(k_2^1 - k_2)]$$

Therefore:

$$\Delta T \geq \beta u'(f(k_1) - (1+n_1)k_2)f'(k_1)(k_1 - k_1^1)+$$
$$\beta u'(f(k_1) - (1+n_1)k_2)f'(k_1)(k_1 - k_1^1)+$$
$$\beta u'(f(k_1) - (1+n_1)k_2)(1+n_1)(k_2^1 - k_2)+$$
$$\beta^2[u(f(k_2) - (1+n_2)k_3) - u(f(k_2^1) - (1+n_2)k_3^1)]+$$
$$\ldots + \beta^T[u(f(k_T) - (1+n_T)k_{T+1}) - u(f(k_T^1) - (1+n_T)k_{T+1}^1)]$$

Note that the first two term of the sum canceled out, and repeating the procedure we obtain:

$$\Delta T \geq \beta^T u'(f(k_T - (1+n_T)k_{T+1})(1+n_T)(k_{T+1}^1 - k_{T+1}) \geq$$
$$-\beta^T u'(f(k_T - (1+n_T)k_{T+1})(1+n_T)k_{T+1}$$

Computing the limit with $T \to +\infty$, we have that $\Delta T \geq 0$, which implies that:

$$\sum_{t=0}^{+\infty} \beta^t u(f(k_t) - (1+n_t)k_{t+1}) \geq \sum_{t=0}^{+\infty} \beta^t u(f(k_t^1) - (1+n_t)k_{t+1}^1)$$

Then, the present value of the utility is higher when the sequence is $\{k_t\}_{t=0}$ than when we consider other feasible sequence. This implies that the sequence is optimal.

5.6 Proof of Theorem 1

Proof. We will prove that

$$a_t = \log(1+n_{t-1}) + \alpha\log(1+n_{t-2}) + \ldots + \alpha^{t-1}\log(1+n_0)$$

converges.

First, it is straightforward to show that:

$$a_{t+1} - \alpha a_t = \log(1+n_t)$$

We have that $n_t \geq n_{t+1}$, for all t, thus: $\log(1+n_t) \geq \log(1+n_{t+1}), \forall t$, which means that $\{\log(1+n_t)\}_{t=0}$ is decreasing, and so it is the sequence $\{a_{t+1} - \alpha a_t\}_{t=0}$. Then $\{a_{t+1} - \alpha a_t\}_{t=0}$ satisfies that:

$$a_{t+1} - \alpha a_t \leq a_t - \alpha a_{t-1}$$

$$a_{t+1} - a_t \leq \alpha(a_t - a_{t-1})$$

But, $a_t - \alpha a_{t-1} \leq a_{t-1} - \alpha a_{t-2}$ implies that $a_t - a_{t-1} \leq \alpha(a_{t-1} - a_{t-2})$; then by induction:

$$a_{t+1} - a_t \leq \alpha^t(a_1 - a_0)$$

1. If $a_1 < a_0$ then $a_1 - a_0 < 0$ and $a_{t+1} - a_t < 0$ since $\alpha \in (0,1)$. This means that: $\{a_t\}_{t=0}$ is decreasing and it is bounded from below since $a_t = \log(1+n_{t-1}) + \alpha\log(1+n_{t-2}) + + \alpha^{t-1}\log(1+n_0) \geq 0$; thus it converges.
2. If for some $T \in N$ it holds that: $a_T \leq a_{T-1}$, then: $a_t \leq a_{t-1}$ for all $t \geq T$. Thus, the sequence converges.
3. Finally, we will show that $\{a_t\}_{t=0}$ cannot be an increasing sequence. Assume, by contradiction, that it is increasing.
   If $a_1 > a_0 = \log(1+n_0)$, that means:

   $$a_1 = \alpha\log(1+n_0) + \log(1+n_1) > \log(1+n_0) = a_0$$

   $$\Rightarrow \frac{\log(1+n_1)}{\log(1+n_0)} > 1 - \alpha$$

   If $a_2 > a_1$, then: $a_2 = \log(1+n_2) + \alpha a_1 > a_1$ and
   $\log(1+n_2) > (1-\alpha)a_1 > (1-\alpha)a_0 = (1-\alpha)\log(1+n_0)$.
   Then, $\frac{\log(1+n_2)}{\log(1+n_0)} > (1-\alpha)$, and by induction, if $\{a_t\}_{t=0}$ is increasing, we have for all t

   $$\frac{\log(1+n_t)}{\log(1+n_0)} > 1 - \alpha > 0$$

   But, $\frac{\log(1+n_t)}{\log(1+n_0)} \to 0$ when $t \to \infty$, which is a contradiction. Therefore, $\{a_t\}_{t=0}$ is not increasing.

Finally, since $a_{t+1} - \alpha a_t = \log(1+n_t)$, by taking limit, it can be shown that a_t tends to 0.

□

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